Noise-induced anomalous diffusion over a periodically modulated saddle

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We study analytically and numerically the anomalous diffusion across periodically modulated parabolic potential within Langevin and Fokker-Planck descriptions. We find that the probability of particles passing over the saddle is affected strikingly by the periodical modulation with average zero bias. Particularly, the initial phase plays an important role in the modulation effect. The effect of the correlation time of external Ornstein-Uhlenbeck noise on dynamical process is also discussed. A reduction in overpassing probability is observed due to finite correlation time.

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I. INTRODUCTION

Ever since Kramers' pioneering work [1] on escape over a potential barrier in dissipative systems, noise-induced transport over a potential barrier has been widely studied in various fields in biology, chemistry, engineering, and physics (see review [2]). Recent years have witnessed a growing interest in diffusion over a saddle [3–7] in nuclear physics in which fusion dynamics of heavy ions is considered as inverse Kramers problem [3,7]. For this problem, one-dimensional Langevin equations satisfying fluctuation-dissipation theorem have been applied to describe the motions of thermally activated particles [4,5]. By assuming the potential around the saddle approximately as quadratic function, exact expressions of stationary probability for particles passing over the saddle have been explicitly obtained. It has been reported that a certain proportion of particles surmount the barrier in the long-time limit, even with moderately small initial kinetic energy [5]. It is notable that in the recent years much work has been devoted to the systems driven by periodically modulated potentials (see review [8]). Examples include parametrically excited oscillators [9,10], escape from a metastable state of periodically modulated systems [11,12], and anomalous diffusion in time-varying potential landscapes [13–15]. Modulation makes potentials varying in time periodically, and thus it becomes more efficient to control the system parameters [9,11,12].

In a more realistic environment, it is necessary to consider a finite correlation time of noise instead of a Dirac delta correlation noise only for mathematically simplicity [16–18]. Such noise (colored noise) has a frequency-dependent spectral density, resulting in non-Markovian dynamics. For the last few decades, the transport induced by equilibrium or nonequilibrium colored noise has attracted a large amount of interest. Within various Langevin and Fokker-Planck descriptions, the static and dynamical properties have been evaluated for small-to-moderate-to-large values of the correlation time and successfully applied to a large variety of systems in physical, biological, and other fields [5,7,16–24].

The aim of this paper is twofold. One is to study anomalous diffusion over a potential barrier periodically varying in time driven by Gaussian white noise. Similar periodical modulation can be found in previous studies such as parametric resonance of optically trapped systems [9] and parametric oscillations [10,14,15]. Here we extend them to the problem of diffusion over a parabolic saddle and investigate modulation effect on the overpassing probability of particles. The other is to explore the stochastic dynamics of the collective relevant degrees of freedom submitted to externally correlated noise. In the present paper, we consider overdamped Langevin equation and corresponding Fokker-Planck equation. As a main result, we find that periodically modulated potential barrier with average zero bias strongly affects the overpassing probability as compared with time-independent case. Another interesting result in this work is a reduction in overpassing probability in systems driven by external colored noise due to finite correlation time of colored noise.

This paper is organized as follows. In Sec II, we introduce a system with parabolic barrier modulated temporally in cosinusoidal form. We derive the explicit expression of the probability passing over the saddle and simulate the motion of particles. In Sec III, we investigate the effects of the external Ornstein-Uhlenbeck noise on the dynamics of the system. We take the integral Euler-Maruyama method to generate the exponentially correlation noise in numerical simulations. The conclusion and the discussion are given in Sec IV.

II. DIFFUSION DRIVEN BY GAUSSIAN WHITE NOISE

In this section we consider the case of motion across a time-periodic parabolic potential barrier in the presence of Gaussian white noise. The dynamics of the particle can be described by one-dimensional Langevin equation [4,7,25],

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \frac{1}{m}\frac{\partial U(x,t)}{\partial x} = \xi(t), \tag{1}$$

where β is the reduced friction coefficient, and $U(x,t) = -f(t)m\Omega^2x^2/2$ with $f(t)$ a time-periodic function. The stochastic force has Gaussian distribution with the first and the second moments

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$$\langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t)\xi(t') \rangle = 2\frac{\beta k_B T}{m}\delta(t-t'), \tag{2}$$

where k_B, T, and m are the Boltzmann constant, the temperature, and the mass of the particle, respectively.

In this paper, we focus on particle diffusion in overdamped case. In the overdamped limit, $\beta \gg \Omega$, Eq. (1) is reduced to

$$\frac{dx}{dt} - f(t)\frac{\Omega^2}{\beta}x = \eta(t), \tag{3}$$

with the properties

$$\langle \eta(t) \rangle = 0 \quad \text{and} \quad \langle \eta(t)\eta(t') \rangle = 2D\delta(t-t'), \tag{4}$$

where $D = k_B T/(m\beta)$ is the intensity of Gaussian white noise $\eta(t)$. One can obtain the position of a particle at time t by integrating Eq. (3) [26],

$$x(t) = x_0 G(t) + G(t)\int_0^t \frac{\eta(s)}{G(s)}ds, \tag{5}$$

where $G(t) = \exp[\frac{\Omega^2}{\beta}\int_0^t f(s)ds]$ and $x_0 < 0$ is initial position of the particle. Thus, the mean value of random variable $x(t)$ reads as

$$\langle x(t) \rangle = x_0 G(t), \tag{6}$$

and the variance of $x(t)$ is

$$\sigma^2(t) = G^2(t)I = G^2(t)\int_0^t\int_0^t \frac{\langle \eta(s)\eta(s') \rangle}{G(s)G(s')}dsds', \tag{7}$$

where the integral I is defined as

$$I = \int_0^t\int_0^t \frac{\langle \eta(s)\eta(s') \rangle}{G(s)G(s')}dsds' = 2D\int_0^t G^{-2}(s)ds. \tag{8}$$

The Fokker-Planck equation corresponding to Eq. (3) can be written as [24–27]

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\left(f(t)\frac{\Omega^2}{\beta}xW\right) + D\frac{\partial^2 W}{\partial x^2}, \tag{9}$$

where the initial condition $W(x, t=0) = \delta(x - x_0)$. In the context of Brownian motion, this Fokker-Planck equation is called the Smoluchowski equation.

As shown in Refs. [25,26], there is no stationary solution of this equation since the drift term on the right side of Eq. (9) is negative. Nevertheless, the linearity of Eq. (3) and Gaussian noise $\eta(t)$ lead necessarily this Smoluchowski equation to a solution of Gaussian distribution [3–7,25]. One can obtain dynamical evolution of probability density function (PDF) $W(x,t)$ via Fourier transform of $W(x,t)$ and subsequent inverse Fourier transform [26]. The PDF reads as

$$W(x,t) = \frac{1}{\sqrt{2\pi}\sigma(t)}\exp\left\{-\frac{[x(t) - \langle x(t) \rangle]^2}{2\sigma^2(t)}\right\}, \tag{10}$$

where $\langle x(t) \rangle$ and $\sigma^2(t)$ are defined as Eqs. (6) and (7), respectively.

Consequently, the overpassing probability that one finds a particle at the other side of the parabolic potential is given by [5–7]

$$P(t) = \int_0^\infty W(x,t)dx = \frac{1}{2}\text{erfc}\left[-\frac{\langle x(t) \rangle}{\sqrt{2}\sigma(t)}\right]. \tag{11}$$

In the overdamped case, it can be reduced to

$$P(t) = \frac{1}{2}\text{erfc}\left[-\frac{x_0}{\sqrt{2I}}\right] \tag{12}$$

by virtue of Eqs. (6)–(8). This indicates that the overpassing probability is totally determined by the integral I for any given initial position x_0 of a particle. If the integral I converges for large times t, the probability has an asymptotic value of

$$P(t \to \infty) = \lim_{t\to\infty}\frac{1}{2}\text{erfc}\left[-\frac{x_0}{\sqrt{2I}}\right] \tag{13}$$

while a divergent I will lead to the maximum value $P = 0.5$ of the overpassing probability.

If the parabolic potential is time independent, for example, $f(t) = 1$, the system has a uniform potential saddle whose dynamics has been extensively studied [3–5]. In this case, probability (12) is reduced to

$$P(t) = \frac{1}{2}\text{erfc}\left[-\frac{x_0}{\sqrt{\frac{2k_B T}{m\Omega^2}(1 - e^{-2\Omega^2 t/\beta})}}\right]. \tag{14}$$

In the long-time limit, this probability converges to

$$P(t \to \infty) = \frac{1}{2}\text{erfc}\left[\sqrt{\frac{B}{k_B T}}\right], \tag{15}$$

where $B = \frac{1}{2}m\Omega^2 x_0^2$ denotes the barrier height measured from the initial position. This recovers the result of Refs. [4,5] as special case of overdamped limit for $\beta/\Omega \gg 1$.

In this work we are interested in the case where the parabolic saddle varies periodically in time with $f(t) = 1 + g\cos(\omega t + \phi_0)$, where $0 \le g \le 1$ measures modulation strength, ω the modulation frequency, and $\phi_0 \in [0, 2\pi]$ the fixed initial phase of the periodic modulation. In this situation, integral (8) can be transformed into

$$I = 2De^{2g\sin\phi_0/(\omega\tau_s)}\int_0^t e^{-2s/\tau_s}e^{-2g\sin(\omega s+\phi_0)/(\omega\tau_s)}ds \tag{16}$$

by introducing a characteristic relaxation time, $\tau_s = \beta/\Omega^2$. In terms of Taylor expansion for $\omega > 2/\tau_s$,

$$e^{-2g\sin(\omega s+\phi_0)/(\omega\tau_s)} = 1 - \frac{2\sin(\omega s + \phi_0)}{\omega\tau_s}g + \frac{2\sin^2(\omega s + \phi_0)}{\omega^2\tau_s^2}g^2 + \mathcal{O}(g^3), \tag{17}$$

substituting Eq. (17) into Eq. (16) and neglecting the term $\mathcal{O}(g^3)$, one can obtain the overpassing probability from Eq. (13) after straightforward calculations,

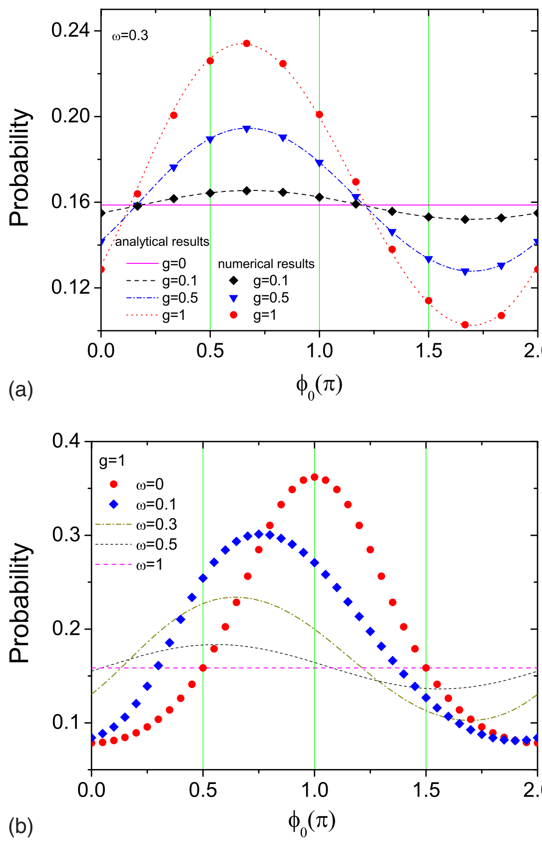

FIG. 1. (Color online) The asymptotic probability for particles passing over time-periodic saddle. The lines correspond to analytical prediction Eq. (18). The parameters are $T=4$, $k_B=1$, $x_0=-2$, $B=2$, $\Omega=1$, and $\tau_s=10$. (a) Probability as a function of ϕ_0 for given modulation frequency $\omega=0.3$. (b) Probability as a function of ϕ_0 for given modulation strength $g=1$. The probabilities for $\omega=0$ (●) and $\omega=0.1$ (♦) are numerical results.

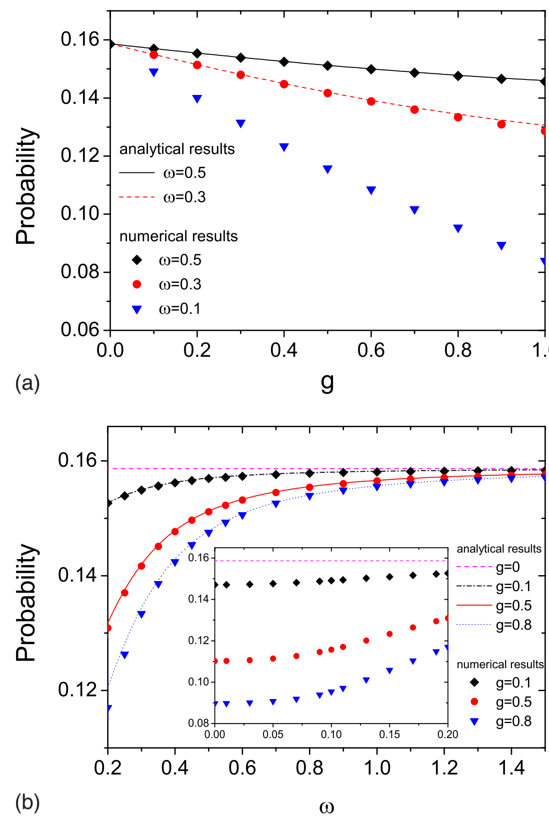

FIG. 2. (Color online) The asymptotic probability for particles passing over periodical modulation saddle. The lines correspond to analytical prediction given by Eq. (18). Here $\phi_0=0$. Other main parameters are same as those in Fig. 1. (a) Probability as a function of modulation strength g. (b) Probability as a function of modulation frequency ω. The probability for small frequency $0<\omega<0.2$ is shown in the inset.

$$P(t\to\infty)=\frac{1}{2}\mathrm{erfc}\left[\sqrt{\frac{B}{k_BT}}e^{-g\sin\phi_0/(\omega\tau_s)}(1+ag+bg^2)^{-1/2}\right], \tag{18}$$

with

$$a=-\frac{4}{4+\omega^2\tau_s^2}\frac{1}{\omega\tau_s}(2\sin\phi_0+\omega\tau_s\cos\phi_0),$$

$$b=\frac{1}{\omega^2\tau_s^2}+\frac{1}{1+\omega^2\tau_s^2}\frac{1}{\omega^2\tau_s^2}(-\cos 2\phi_0+\omega\tau_s\sin 2\phi_0).$$

To demonstrate the modulation effect on diffusion, we plot analytical prediction of asymptotic probability (18) versus initial phase ϕ_0 for various modulation strength g and frequency ω, as shown in Fig. 1. In this paper, Boltzmann constant k_B and the particle mass m are set as unity. The parameters $x_0=-2$, $\Omega=1$ are taken. The temperature is $T=4$ unless otherwise stated. To meet the demand of overdamped limit, $\beta/\Omega=10$ is adopted, and thus, $\tau_s=10$ is obtained.

On the other hand, we integrate numerically Langevin Eq. (3) with Eq. (4) using the standard Euler-Maruyama method with a small time step $\Delta t=0.001$. We simulate the motion of an ensemble of 50 000 particles starting at $x_0=-2$. After a long time, the overpassing probability comes to a steady value. The numerical results of asymptotic probability averaging over 100 realizations are also shown in Fig. 1.

Surprisingly, our study suggests that the diffusion over the time-periodic saddle is greatly dependent on modulation parameters, in which initial phase ϕ_0 of modulation plays a key role. There are three cases where the overpassing probability behaves significantly different for various values of ϕ_0. Case I, in the area $3\pi/2\le\phi_0\le 2\pi$, it is found that the probability is suppressed. Case II, in the area $\pi/2\le\phi_0\le\pi$, the probability is greatly enhanced as a comparison [see Figs. 1(a) and 1(b)]. Case III, $0<\phi_0<\pi/2$ and $\pi<\phi_0<3\pi/2$, whether the probability is suppressed or enhanced depends on the parameter choice of modulation strength g and frequency ω. As shown in Fig. 1(b), one can see a transition from suppression to enhancement of probability with growing ω at $\phi_0=\pi/4$. While a transition of enhancement to suppression is observed at $\phi_0=5\pi/4$. Notably, all modulation effects die down for small g and large ω.

In the following we focus on the case of $\phi_0=0$ to make a further investigation. As shown in Fig. 2, the overpassing probability is suppressed significantly due to the periodical modulation in this case. Starting from the same value of $P=0.1586$ for $g=0$, the probability decreases monotonously

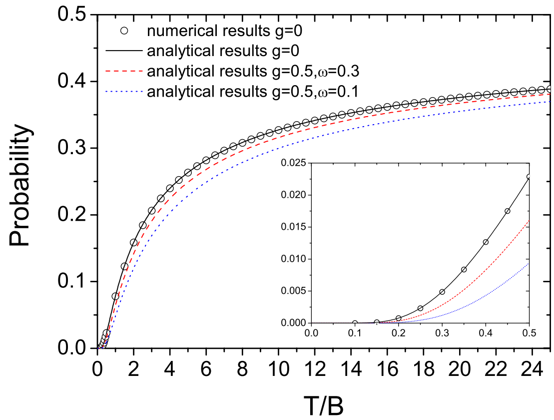

FIG. 3. (Color online) Temperature dependence of the overpassing probability. Here a dimensionless temperature T/B is taken with $B=2$ the barrier height measured from the initial position. The lines are analytical probability (18) with $\phi_0=0$. The details for low temperatures are shown in the inset.

with the increasing of g at different modulation frequencies ω, as shown in Fig. 2(a). The modulation frequency dependence of probability is shown by lines in Fig. 2(b). The probability increases with growing frequency, and asymptotically close to the value of probability at $g=0$ in the high-frequency limit.

The numerical results of asymptotic probability can be found in Figs. 2(a) and 2(b), which are in excellent agreement with analytical result (18). Notably, we obtain the overpassing probability within a wide range of frequency by numerical simulations, even for low modulation frequency $\omega(0\leq\omega\leq 2/\tau_s)$, as shown in the inset of Fig. 2(b). We find that both large modulation strength g and low modulation frequency ω suppress the probability by a large margin. This suppression is weakened for higher frequency, indicating that only low-frequency modulation affects significantly diffusion over the time-periodic potential saddle.

Temperature dependence of the overpassing probability is depicted in Fig. 3. Both analytical formula (18) and numerical results have verified previous findings that large strength of noise is helpful for particles to pass over the static parabolic saddle.

The time evolution of the overpassing probability is given in Fig. 4(a). The normalized PDF at time $t=40$ and $t=70$ are demonstrated in Figs. 4(b) and 4(c), respectively, corresponding to steady overpassing probability for each case. It is clear that only a small number of particles can surmount the saddle in the modulated system as compared with the case of static potential saddle.

III. DIFFUSION DRIVEN BY GAUSSIAN COLORED NOISE

It is a more frequent case that real fluctuations of the random force are correlated. In this section, we discuss dynamical process driven by exponentially correlated Gaussian (Ornstein-Uhlenbeck) noise arising from external fluctuation [16–18]. The system is described by overdamped Langevin equation

$$\frac{dx}{dt} - f(t)\frac{\Omega^2}{\beta}x = \epsilon(t), \tag{19}$$

where $\beta/\Omega \gg 1$. In this section, we take $f(t)=1+g\cos(\omega t)$ for simplicity. The stationary noise $\epsilon(t)$ satisfies

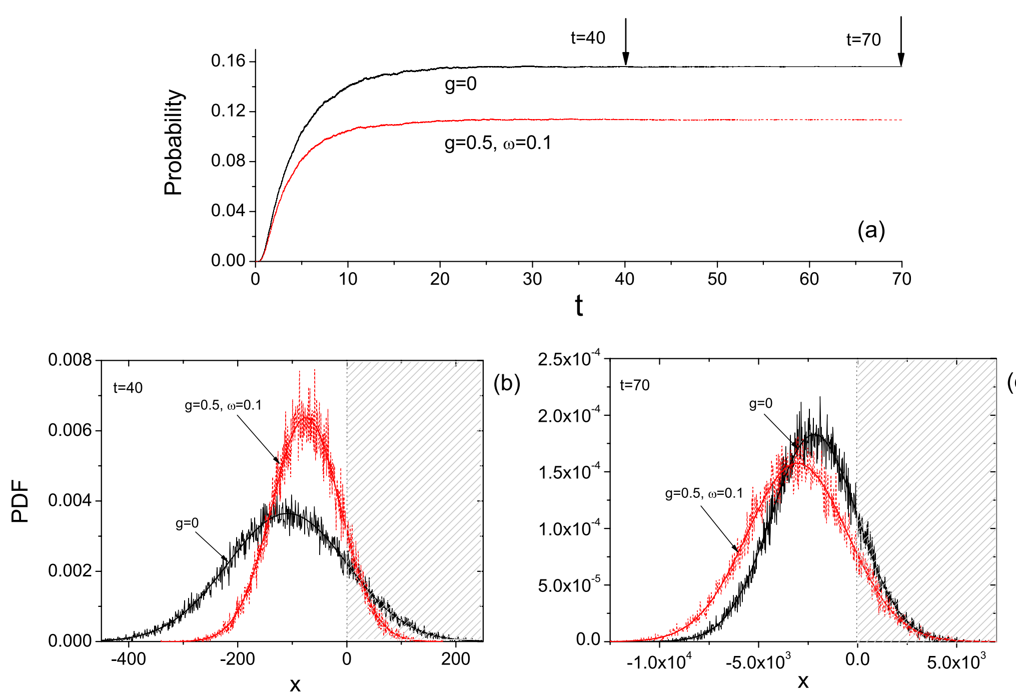

FIG. 4. (Color online) (a) Numerical results of time evolution of overpassing probability. Here $\phi_0=0$. The other modulation parameters are $g=0$ and $g=0.5$, $\omega=0.1$, respectively. Numerical PDFs and corresponding Gaussian fit lines at time $t=40$ and $t=70$ are shown in (b) and (c), respectively. The overpassing probability can be seen in the dark area.

$$\langle \epsilon(t)\rangle = 0 \text{ and } \langle \epsilon(t)\epsilon(t')\rangle = \frac{D_0}{\tau_c}\exp\left(\frac{-|t-t'|}{\tau_c}\right), \qquad (20)$$

where D_0 is the intensity and τ_c is the correlation time of the colored noise.

Thus, there are two characteristic times, relaxation time $\tau_s=\beta/\Omega^2$ and correlated time τ_c of colored noise in such a system. When the correlation time is much less than the relaxation time of the system, say $\tau_c \ll \tau_s$, Gaussian white noise with statistic properties [Eq. (4)] is revisited. And then particles behave like those discussed in Sec. II. If τ_c cannot be neglected, the dynamical process is non-Markovian that the system has memory effect of previous state through the colored noise [16]. In this situation, the mean value $\langle x(t)\rangle$ in Eq. (6) is still valid while the variance $\sigma^2(t)$ becomes

$$\sigma^2(t) = G^2(t)I = G^2(t)\int_0^t\int_0^t \frac{\langle \epsilon(s)\epsilon(s')\rangle}{G(s)G(s')}ds ds', \qquad (21)$$

where the integral I reads as

$$I = \int_0^t\int_0^t \frac{\langle \epsilon(s)\epsilon(s')\rangle}{G(s)G(s')}ds ds'. \qquad (22)$$

First, we discuss diffusion over a uniform saddle driven by Gaussian colored noise $\epsilon(t)$. When $g=0$, the integral has explicit form in the long-time limit (see Appendix)

$$I(t\to\infty) = D_0\tau_s\left(\frac{1}{1+\tau}\right) \qquad (23)$$

in terms of a dimensionless time $\tau=\tau_c/\tau_s$. This implies that the diffusion is greatly slowed down in the presence of colored noise with finite correlation time.

Due to colored noise, ordinary Fokker-Planck Eq. (9) cannot be applied to describe time evolution of PDF $W(x,t)$ of such a system. The exact equation for the rate of change of $W(x,t)$ has been given by Hänggi *et al.* within a master-equation description [16,28]. The solution is Gaussian, non-Markovian in terms of initial probability distribution $W(x,0)=\delta(x-x_0)$ [16],

$$W(x,t) = \frac{1}{\sqrt{2\pi\alpha(t)}}\exp\left\{-\frac{[x-\gamma(t)]^2}{2\alpha(t)}\right\}, \qquad (24)$$

with

$$\alpha(t) = 2\int_0^t D'(s)e^{2(t-s)/\tau_s}ds,$$

$$\gamma(t) = x_0 e^{t/\tau_s},$$

where time-dependent diffusion coefficient is written as

$$D'(t) = \int_0^t \langle \epsilon(t)\epsilon(s)\rangle e^{(t-s)/\tau_s}ds. \qquad (25)$$

For exponentially correlated noise (20), we have

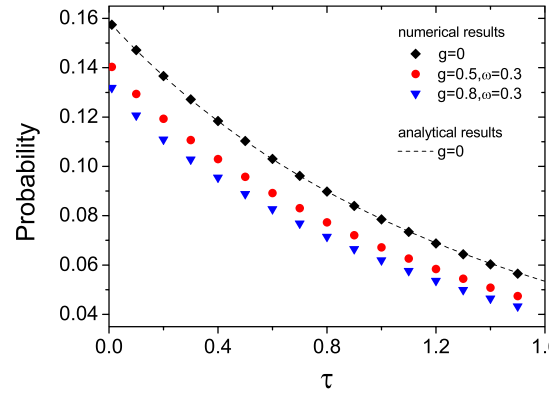

FIG. 5. (Color online) The overpassing probability as a function of correlation time τ. Here $B=2$, $D_0=0.4$, and $\omega=0.3$. The dashed line is the analytical prediction of Eq. (27).

$$D'(t) = \frac{D_0}{1-\tau}[1-e^{(1/\tau_s-1/\tau_c)t}], \qquad (26)$$

which keeps positive for different values of τ_c. After some straightforward calculations, one can find that $\gamma(t)$ and $\alpha(t)$ just correspond to the dynamical evolution of mean position of the particles $\langle x(t)\rangle$ and the variance $\sigma^2(t)$ in Eqs. (6) and (21), respectively.

Integrating Eq. (24) from $x=0$ to $x\to\infty$, one obtains explicitly the asymptotic overpassing probability

$$P(t\to\infty) = \frac{1}{2}\mathrm{erfc}\left[\sqrt{\frac{B(1+\tau)}{m\beta D_0}}\right]. \qquad (27)$$

This indicates that the finite correlation time τ_c will lead to the decreasing of overpassing probability, which is the other main point of this work. Notice that in this work we discuss the motion of particles driven by external colored noise, while the systems driven by thermal colored noise have been studied in Refs. [5,7]. where fluctuation-dissipation theorem is satisfied. In Fig. 5 the dependence of probability on the dimensionless time τ is shown according to Eq. (27).

In the following, we concentrate our attention to numerical simulation of Langevin Eq. (19) with the colored noise (20) to verify this analytical prediction. We generate the exponentially correlated colored noise from Gaussian white noise. To aim at this, we rewrite Langevin Eq. (19) into two-component form [18,29],

$$\frac{dx}{dt} = [1+g\cos(\omega t)]\frac{\Omega^2}{\beta}x + \epsilon(t), \qquad (28a)$$

$$\frac{d\epsilon(t)}{dt} = -\frac{1}{\tau_c}\epsilon(t) + \frac{1}{\tau_c}\eta(t). \qquad (28b)$$

where $\eta(t)$ is still Gaussian white noise with the properties [Eq. (4)] and thus $\epsilon(t)$ is the exponentially correlated noise possessing the properties [Eq. (20)].

We integrate numerically the pair of Langevin Eqs. (28a) and (28b) using the integral Euler-Maruyama algorithm proposed by Fox *et al.* [29]. We take the step size $\Delta t=0.001$ and an ensemble of 50 000 particles in our simulations.

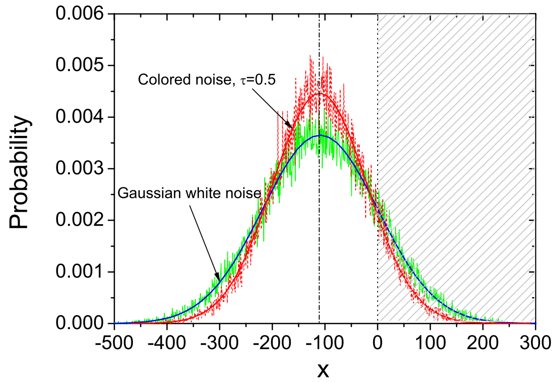

FIG. 6. (Color online) The simulated PDFs and corresponding Gaussian fit lines at time $t=40$ submitted to Gaussian white noise and Gaussian colored noise respectively. $\tau=\tau_c/\tau_s=0.5$. The overpassing probability can be found in the dark area.

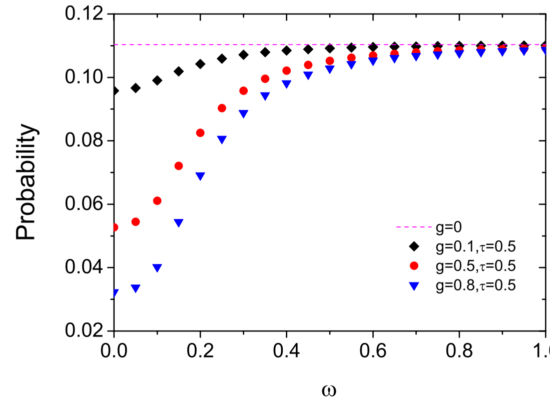

FIG. 7. (Color online) The numerical results of overpassing probability as a function of modulation frequency ω in the system submitted to colored noise. $\tau=0.5$ and $D_0=0.4$ in the simulations. The analytical result of Eq. (27) for $\tau=0.5$ is shown (dashed line) to guide the eyes.

For the case of $g=0$, we calculate extensively the overpassing probability for different correlation times τ_c in our simulations. A decaying probability is observed from small-to-moderate-to-large dimensionless time τ, as shown in Fig. 5, which is consistent with analytical result (27).

To study the dynamics of such system, we simulate an ensemble of particles starting from $x_0=-2$ driven by Gaussian colored noise with $\tau=0.5$ and an ensemble of particles driven by Gaussian white noise as comparison. The numerical results of time evolution of PDF in two systems are shown in Fig. 6. One can see clearly that the centers of two Gaussian PDFs move synchronously, while the variance of the former is suppressed due to the finite correlation time τ_c. Therefore, there is no surprise of a suppressed probability for particles passing over the saddle for the case of colored noise driving.

Now we discuss the case of $0<g\leq 1$. As was expected, a superimposed effect on the diffusion process arising from the combination of finite correlation time and periodical modulation is observed in our numerical simulations. The asymptotic probability as a function of correlation time τ for different g and ω is shown in Fig. 5. Besides, the frequency dependence of the overpassing probability can be found in Fig. 7. The numerical results show that the probability is suppressed by large modulation strength g and low frequency ω, and it is insusceptible to the periodical modulation for high frequency. This is similar to those in the system driven by Gaussian white noise.

IV. DISCUSSION AND CONCLUSION

In the present paper, we have investigated analytically and numerically anomalous diffusion over periodical modulated parabolic saddle by means of Langevin and Fokker-Planck equations. We have demonstrated that the periodical modulation saddle strongly affect the probability of particles passing over the parabolic saddle. The overpassing probability has totally different behaviors for various values of ϕ_0. As an example, we found the probability is suppressed for time-periodic modulation with $f(t)=1+g\cos(\omega t)$, while is enhanced with $f(t)=1-g\cos(\omega t)$. An alternate suppression and enhancement effect is observed for $\phi_0=\pi/4$ and $\phi_0=5\pi/4$, which is greatly dependent on the modulation frequency ω. Notably, both small modulation strength g and high modulation ω will weaken these modulation effects, and hence the results of diffusion over static saddle are recovered.

For the case driven by the external colored noise, the dynamical process is non-Markovian, and then the diffusion is greatly influenced by correlation time τ_c. If $\tau_c\to 0$, the process in Gaussian white noise is recurred again. If τ_c cannot be neglected, the overpassing probability is suppressed by the external colored noise due to finite correlated time.

In this present paper, we consider the motion of particles by the overdamped Langevin equation with unbounded boundary condition, as was done in Refs [3–7]. We have calculated numerically that the particle number of passing over the saddle from one side to the other decays exponentially with time. That is to say, there is almost no particle passing over the saddle in the long-time limit, leading to a small probability of a particle returning to its initial position. Hence, the problem in this work is equivalent (in the long-time limit) to the first passage time problem. More problems of escape out of the potential varying in time determinately and stochastically deserve further intensive study in the future.

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APPENDIX

Starting from Langevin Eq. (19) with property (20), integral (22) is transformed to

$$I=\frac{D_0}{\tau_c}\int_0^t\int_0^t e^{-(s+s')/\tau_s}e^{-|s-s'|/\tau_c}ds ds' \tag{A1}$$

for $g=0$. By introducing the variables $Q=s+s'$ and $q=s-s'$, one can obtain

$$I=\frac{D_0}{2\tau_c}\left[\int_0^t e^{-Q/\tau_s}dQ\int_{-Q}^{Q}e^{-|q|/\tau_c}dq + \int_t^{2t}e^{-Q/\tau_s}dQ\int_{Q-2t}^{2t-Q}e^{-|q|/\tau_c}dq\right]. \tag{A2}$$

After straightforward calculations, we have

$$I=D_0\tau_s\left(\frac{1}{1+\tau}\right)+\frac{2D_0\tau_c}{(1+\tau)(1-\tau)}e^{-(1/\tau_s+1/\tau_c)t}-\frac{D_0\tau_s}{(1-\tau)}e^{-2t/\tau_s}. \tag{A3}$$

In the long-time limit, integral (22) becomes

$$I=D_0\tau_s\left(\frac{1}{1+\tau}\right). \tag{A4}$$

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